Wavefront Construction: A Differential Geometry Approach

S. Hildebrand, C. Aprea, and M. Fehler, Los Alamos Seismic Research Center, Los Alamos National Laboratory.*

Summary

A ray travel time wavefront represents a mathematical object called a manifold. In particular, paraxial ray theory contains elements that predict the Gaussian curvature of the wavefront surface. There are locations on this wavefront where the curvature is singular; in geometric optics terms, these are the ray caustic regions. These singular points or lines represent virtual sources. As such, the ray density in the neighborhood of these wavefront surface singularities must be large to model such sources. Conversely in regions of small curvature, the ray density can be low and still adequately represent the wavefront shape. These observations lead us to suggest new ray addition criteria for the wavefront construction algorithm: *the distance between two neighboring rays must be less than the radius of wavefront curvature between these rays*. This paraxial ray approximation insures that the angle between the neighboring ray slowness vectors is bounded and that the paraxial ray travel time estimate is valid for this interval. For velocity models where the projection of the wavefront upon the receiver plane is multi-valued, the paraxial ray method yields an elegant procedure to interpolate the resulting travel time and complex amplitude fields.

Introduction

Wavefront construction ray tracing offers some of the advantages of modeling wavefield phenomena but maintains the speed of ray methods. In wavefront construction, rays are initiated at a source as in conventional ray tracing and they are extrapolated through the model in equal intervals of time; however, it is assumed that each ray is connected to its neighbors by a piece of a "wavefront" and the entire ray field defines a wavefront manifold. The relation between the rays and the wavefronts allows new rays to be added when rays become widely separated. Mutiple arrivals can be modeled and amplitudes can be estimated from simple geometrical considerations. In addition, phase terms, which are the result of crossing raypaths, can be calculated and used in migrations. The advantage of wavefront construction over standard ray shooting methods for calculating traveltimes are that the computational cost is significantly less in wavefront construction. Fewer rays need to be calculated in complex models to obtain a fixed ray density. We have a new version of a 2-D wavefront construction algorithm that uses the wavefront curvature and distance between neighboring rays as criteria to interpolate new rays on the wavefront.

The wavefront construction algorithm consists of two parts. In the first part, a series of rays are traced for a finite time step. In the second part of the algorithm, a decision is made where to add rays to the wavefront. This is done in a purely geometric manner. The standard criteria used are two tests (Vinje, et al., 1993a and 1993b). The first test is to determine if the distance between neighboring rays is beyond some threshold distance. The second test is to check if the angle between the two neighboring rays is larger than some threshold angle. These two tests, distance and angle between neighboring rays, assumes that the wavefront surface is smooth over some minimum distance.

Lambare, et al. (1996) have suggested another criterion to insure the wavefront has a certain ray density. They assumed that the traveltime error between neighboring rays should be sufficiently small that one can use linear interpolation between them. This criterion is good for forward modeling purposes, but could lead to increased ray tracing and computational costs to satisfy the linear traveltime interpolation constraint. Since our objective is to obtain a ray method that can be used to determine migration traveltimes, we would like to minimize the number of rays that must be traced while having sufficient numbers to accurately predict the complex wavefront surface. To accomplish this, we suggest using the ratio of the distance between neighboring rays and the ray wavefront curvature as the criterion to add a ray to the wavefront.

The paraxial ray approximation allows one to predict the traveltime of a neighboring ray. It assumes that the distance between the rays is smaller than the local radius of wavefront curvature. This assumption also implies that the angle between the rays is small. These paraxial ray assumptions are similar to the standard wavefront construction test set; but, these conditions are slowness model dependent via the ray calculation of radius of wavefront curvature. In addition, paraxial ray approximation allows one to use a higher order quadratic equation to predict the ray travel time manifold/wavefront. In the next section, paraxial ray theory is discussed in terms of a wavefront manifold geometry.

Wavefront Construction: A Differential Geometry Approach

Theory

The paraxial approximation for a neighboring ray travel time can be represented by the following Taylor series expansion:

$$t(\mathbf{x} + \delta\mathbf{x}) = t(\mathbf{x}) + \mathbf{p} \bullet \delta\mathbf{x} + 1/2\ \delta\mathbf{x}^T \mathbf{M}\ \delta\mathbf{x}$$

where $\mathbf{p} = \nabla t$ is the ray slowness vector and $\mathbf{M} = \nabla \nabla t$ is the second time derivative operator.

A local orthogonal reference frame attached to the wavefront is called the ray centered coordinate system. This frame has the following unit vector basis set:

$\mathbf{e} = v\,\mathbf{p}$ unit vector in ray's direction,
$\mathbf{n}$ unit normal vector $\mathbf{n} \bullet \mathbf{e} = 0$, and
$\mathbf{b}$ unit binormal vector $\mathbf{n} \times \mathbf{b} = \mathbf{e}$.

Unitary transform to ray centered coordinate system of the second time derivative operator yields (White, et al., 1987)

$$[\mathbf{e}, \mathbf{n}, \mathbf{b}]^T\ \mathbf{M}\ [\mathbf{e}, \mathbf{n}, \mathbf{b}] = \begin{vmatrix} \mathbf{e}^T\mathbf{M\,e} & \mathbf{e}^T\mathbf{M\,n} & \mathbf{e}^T\mathbf{M\,b} \\ \mathbf{n}^T\mathbf{M\,e} & \mathbf{n}^T\mathbf{M\,n} & \mathbf{n}^T\mathbf{M\,b} \\ \mathbf{b}^T\mathbf{M\,e} & \mathbf{b}^T\mathbf{M\,n} & \mathbf{b}^T\mathbf{M\,b} \end{vmatrix}$$

with

$$\mathbf{e}^T\mathbf{M\,e} = 1/(2\,s)\,\mathbf{e}^T\ \nabla s^2$$
$$\mathbf{n}^T\mathbf{M\,e} = 1/(2\,s)\,\mathbf{n}^T\ \nabla s^2$$
$$\mathbf{b}^T\mathbf{M\,e} = 1/(2\,s)\,\mathbf{b}^T\ \nabla s^2$$

and the wavefront curvature operator is defined as

$$s\,\mathbf{K} = \begin{vmatrix} \mathbf{n}^T\mathbf{M\,n} & \mathbf{n}^T\mathbf{M\,b} \\ \mathbf{b}^T\mathbf{M\,n} & \mathbf{b}^T\mathbf{M\,b} \end{vmatrix}$$

with eigenvalues which are the reciprocal of the principal radii of wavefront curvature, r1 and r2.

Using the principal radii of wavefront curvature, we can describe some simple wavefront surface patches. If r1 and r2 are equal to infinity, we have a plane wavefront. For the case where r1 and r2 have the same sign, but different magnitude, the wavefront is a synclastic surface, e.g. ellipsoidal shell. If the principal radii have different signs, the surface resembles a saddle, an anclastic surface.

We now have the tools to compute the ray amplitude and phase intercept terms. We start by noting that the transport equation for the ray amplitude, A, can be written as a divergence of a vector:

$$2\,\nabla A \bullet \nabla t + A\,\nabla^2 t = 0$$

or

$$\nabla \bullet (A^2\,\mathbf{p}) = 0$$

This divergence equation can be expressed as a volume integral over a ray tube. Then it can be converted to a surface integral over the ray tube's surface with the only contribution coming from the ends of the ray tube:

$$\int d\Gamma_1\,(A^2\,\mathbf{p} \bullet \mathbf{e}) = \int d\Gamma_2\,(A^2\,\mathbf{p} \bullet \mathbf{e})$$

Since the divergence is equal to zero, this flux term, amplitude squared times ray slowness vector, is conserved. This is consistent with a high frequency approximation that was used to obtain the transport equation, that is, there is no scattering of the energy outside of the ray tube.

Using the radius of curvature parameters, we can write down an expression for the differential surface area which is the product of the differential arc lengths in the perpendicular directions. The differential surface is

$$d\Gamma_1 = (r1\,d\o)\,(r2\,d\beta) = d\Omega/\det(\mathbf{K})$$

where $d\Omega$ is the ray tube solid angle and det is the determinant of the operator. The second differential area is similar to the initial, but with a cosine correction term for ray bending. For ray consistency assumptions, the magnitude of this term is effectively, unity, and is neglected in most formulations. This leads us to the following equation for the amplitude of the ray tube as shown in the equation for geometric spreading:

$$(A^2\,\mathbf{p} \bullet \mathbf{e} / \det(\mathbf{K}))_1 = (A^2\,\mathbf{p} \bullet \mathbf{e} / \det(\mathbf{K}))_2\ \mathbf{e}(t_1) \bullet \mathbf{e}(t_2)$$

Therefore amplitude at the end points of the ray tube are related to the square root of the determinant of the wavefront curvature operator. Because this is asymptotic ray theory, the amplitude goes to infinity as the determinant goes to zero. These are the caustics of the ray where r1 or r2 or both equal zero. If one of the principal radii of wavefront curvature is zero, we have a line singularity, where both are zero, this is a point singularity. To determine the correct phase intercept for these conditions, we use the following equation (Snieder and Lomax, 1996):

$$\det(\mathbf{K})^{1/2} = |\det(\mathbf{K})|^{1/2}\ \exp[\,i\,\pi/2 - i\,\mathrm{sgn}(\mathbf{K})\pi/4]$$

where the function sgn is the number of positive eigenvalues minus the number of negative eigenvalues of the operator. Therefore for a line singularity, we obtain a ninety degree phase shift; and for a point singularity, there is a 180 degree phase shift.

Examples

The wavefront construction algorithm using the radius of wavefront curvature ray addition criterion is tested with two velocity models. One is a 2-D Gaussian velocity model and the other is a 2-D section from the SEG/EAEG salt model.

Wavefront Construction: A Differential Geometry Approach

The 2-D Gaussian velocity model produces a double ray caustic; that is, the radius of wavefront curvature changes sign twice through the low velocity zone. Wavefronts are displayed with the model in Figure 1.

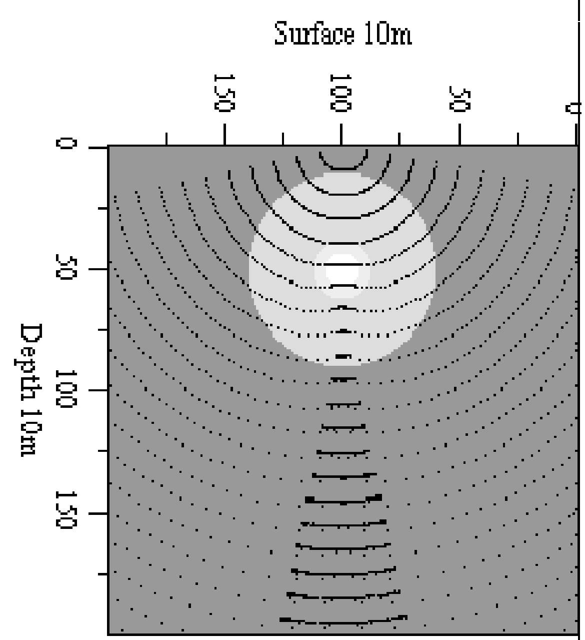

Figure 1: Wavefronts for Gaussian velocity model, $V = 2 - 0.4e^{-.01(x-100)^2 - .01(z-50)^2}$ km/s. Wavefronts displayed every 50 ms.

Receiver Surface

0 20.8 Km

Depth 3.7 Km

Figure 2: Wavefronts for the SEG/EAEG salt velocity model. Wavefronts displayed every 100 milliseconds.

Shown in Figure 2 is a set of wavefronts produced by a buried directional source in a 2D slice of the SEG/EAEG salt model. This is the 2D slice discussed by O'Brien and Gray (1996). Note the series of triplication events due to the high velocity salt body.

Observations

For an isotropic acoustic medium, there are two wavefront manifolds, one in ray position space and the other in ray slowness space. The manifold in ray position space has a surface normal parallel to the slowness vector; the dual to the ray position manifold is the ray slowness manifold with a surface normal parallel to the velocity gradient. Discontinuities in this surface represent wave phenomena such as diffraction and shadow zone illumination that are not predicted by asymptotic ray theory. Therefore an equivalent ray addition rule to account for the ray slowness discontinuities is suggested.

Conclusions

The geometry of a wavefront dictates where the ray field density must be large to adequately model the wave propagation. It is at the point where the curvature becomes large that the number of rays must be high. We have thus suggested a simple ray addition criterion for optimal wavefront construction.

Acknowledgments

We would like to thank the sponsors of the Kirchhoff Migration Algorithm Research Project. This work is funded by the United States Department of Energy through contract W-7405-ENG-36 to the Los Alamos National Laboratory DOE.

References

Lambare, G., Lucio, P. S., and Hanyga, A., 1996, Two-dimensional multivalued traveltime and amplitude maps by uniform sampling of ray field: Geophys. J. Int., **125**, 584-598.

O'Brien, M. J. and Gray, S. H., 1996, Can we image beneath salt?: The Leading Edge, **15**, no. 1, 17-22.

Snieder, R., and Lomax, A., 1996, Wavefield smoothing and the effect of rough velocity perturbations on arrival times and amplitudes, Geophys. J. Int., **125**, 796-812

Thomson and Chapman, 1985, an introduction to Maslov's asymptotic method, Geophys. J.R. astr. Soc., 83, 143-168.

Vinje, V., Iversen, E., and Gjystdal, H., 1993a, Traveltime and amplitude estimation using wavefront construction: Geophysics, **58**, 1157-1166.

Vinje, V., Iversen, E., and Gjystdal, H., 1993b, Estimation of multivalued arrivals in 3D models using wavefront construction: 55th Annual EAEG Meeting, Eur. Ass. Expl. Geophys., Extended Abstracts.

White, B.S., Norris, A., Bayliss, A., and Burridge, R., 1987, Some remarks on the Gaussian beam summation method, Geophys. J.R. astr, 89, 579-636.